

ON MINIMAL AND MAXIMAL p -OPERATOR SPACE STRUCTURES

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ABSTRACT. We show that $L^\infty(\mu)$, in its capacity as multiplication operators on $L^p(\mu)$, is minimal as a p -operator space for a decomposable measure μ . We conclude that $L^1(\mu)$ has a certain maximal type p -operator space structure which facilitates computations with $L^1(\mu)$ and the projective tensor product.

In the theory of operator spaces, there are extremal operator space structures which can be assigned to any Banach space. These arose in the papers [3, 6] and are exposed in the monograph [7]. They have particular value when understanding mappings and tensor products.

In the present article we examine minimal and maximal p -operator space structures. These structures' existences were noted in [10], where they were used to characterise certain algebras as algebras of operators on \mathcal{SQ}_p -spaces. Our primary motivation is to gain the isometric tensor product formula $L^1(\mu) \widehat{\otimes}^p \mathcal{V} \cong L^1(\mu, \mathcal{V})$ for the p -operator projective tensor product of [5]. Here $L^1(\mu)$ has a certain maximal operator space structure, which appears naturally via the embedding of $L^1(\mu) \hookrightarrow L^\infty(\mu)^*$, where $L^\infty(\mu)$ acts on $L^p(\mu)$ as multiplication operators. This is a less obvious task than we had initially hoped, and seems worth an exposition in its own right. The techniques of this article are all classical and elementary.

0.1. Background. Let $1 < p < \infty$, and p' denote the conjugate index so $\frac{1}{p} + \frac{1}{p'} = 1$. The theory of p -operator spaces is designed to give an analogue to the theory of operator spaces on a Hilbert space, which we might call 2-operator spaces. The theory of p -operator spaces has its origins in [13, 12], and was studied extensively in [10]. Daws ([5]) presents these spaces in the format we are using, a format also used extensively by An, Lee and Ruan ([1]). We closely follow the presentation of [5] and use some concepts from [1].

We let $\ell^p(n)$ denote \mathbb{C}^n with the ℓ^p -norm. Given a Banach space \mathcal{V} , a p -operator space structure on \mathcal{V} is a sequence of norms $\|\cdot\|_n$, each norm on $n \times n$ -matrices with entries in \mathcal{V} , which satisfy the axioms below.

- (D_∞) For u in $\mathbb{M}_n(\mathcal{V})$ and v in $\mathbb{M}_m(\mathcal{V})$, $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$.
- (M_p) For u in $\mathbb{M}_n(\mathcal{V})$ and α, β in $\mathbb{M}_n \cong \mathcal{B}(\ell^p(n))$, $\|\alpha u \beta\|_n \leq \|\alpha\|_{\mathcal{B}(\ell^p(n))} \|u\|_n \|\beta\|_{\mathcal{B}(\ell^p(n))}$.

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A Banach space \mathcal{V} , equipped with a sequence of norms as above, will be called a *p-operator space*. In the sequel we will drop the subscript n from the norm on $\mathbb{M}_n(\mathcal{V})$. A linear map $T : \mathcal{V} \rightarrow \mathcal{W}$ gives rise to amplifications $T^{(n)} : \mathbb{M}_n(\mathcal{V}) \rightarrow \mathbb{M}_n(\mathcal{W})$, $T^{(n)}[v_{ij}] = [Tv_{ij}]$. Such a map is called *completely bounded* if $\|T\|_{\text{pcb}} = \sup_n \|T^{(n)}\| < \infty$. Moreover it is called *completely contractive* if $\|T\|_{\text{pcb}} \leq 1$ and a *complete isometry* if each $T^{(n)}$ is an isometry. The space of such maps will be denoted $\mathcal{CB}_p(\mathcal{V}, \mathcal{W})$.

We say a Banach space E is in the class \mathcal{SQ}_p if it is a quotient of a subspace of $L^p(\phi)$ for some measure ϕ . The space $\mathcal{B}(E)$ is a *p-operator space* given identifications $\mathbb{M}_n(\mathcal{B}(E)) \cong \mathcal{B}(\ell^p(n) \otimes^p E) \cong \mathcal{B}(\ell^p(n, E))$. Here $L^p(\phi) \otimes^p E$ is the completion with respect to the norm given by embedding $L^p(\phi) \otimes E \hookrightarrow L^p(\phi, E)$. Moreover, any *p-operator space* admits a complete isometry into $\mathcal{B}(E)$ for some E in \mathcal{SQ}_p ([13, 12]). Spaces which admit complete isometries into $\mathcal{B}(L^p(\phi))$ will admit better properties than general *p-operator spaces*. We will follow [1] and say that such spaces *act on (some) L^p* .

We follow [5] on assigning *p-operator space* structures to mapping spaces. We identify $\mathbb{M}_n(\mathcal{CB}_p(\mathcal{V}, \mathcal{W})) \cong \mathcal{CB}_p(\mathcal{V}, \mathbb{M}_n(\mathcal{W}))$, where $\mathbb{M}_n(\mathcal{W})$ is a *p-operator space* via the identifications $\mathbb{M}_m(\mathbb{M}_n(\mathcal{W})) \cong \mathbb{M}_{mn}(\mathcal{W})$. In particular, for the dual space, $\mathbb{M}_n(\mathcal{V}^*) \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{B}(\ell^p(n)))$, completely isometrically. We have *p-version* of the projective tensor product \otimes^γ and the injective tensor product \otimes^λ ; namely the *p-projective tensor product* $\widehat{\otimes}^p$ of [5] and the *p-injective tensor product* $\check{\otimes}^p$ of [1]. The *p-projective tensor product* enjoys all of the usual functorial properties which are analogues of \otimes^γ , while the theory of $\check{\otimes}^p$ is not as well understood. However, we do have that $\mathbb{M}_n(\mathcal{V}) \cong \mathcal{V} \check{\otimes}^p \mathcal{B}(\ell^p(n))$ completely isometrically.

As observed in [10, p. 89], for a *p-operator space* \mathcal{V} , the algebraic identification $\mathcal{V} \otimes \mathcal{B}(\ell^p(n)) \cong \mathbb{M}_n(\mathcal{V})$ allows us to view $\|\cdot\|_n$ as a reasonable cross-norm on $\mathcal{V} \otimes \mathcal{B}(\ell^p(n))$; see the terminology in [14], for example. Indeed, an application of (M_p) then of (D_∞) shows that $\|[\alpha_{ij}v]\|_n \leq \|\alpha\|_{\mathcal{B}(\ell^p(n))} \|v\|$ for α in \mathbb{M}_n and v in \mathcal{V} ; while [5, Lem. 4.2] — that contractive linear functions are automatically completely contractive — shows that $|\varphi \otimes \psi(v)| \leq \|\varphi\|_{\mathcal{V}^*} \|\psi\|_{\mathcal{B}(\ell^p(n))^*} \|v\|_n$ for any φ and ψ where $v \in \mathcal{V} \otimes \mathcal{B}(\ell^p(n))$. Moreover, if \mathcal{X} is any Banach space, then the algebraic identifications

$$(0.1) \quad \mathbb{M}_n(\mathcal{X}) \cong \mathcal{X} \otimes^\lambda \mathcal{B}(\ell^p(n))$$

(injective tensor product on the right), are easily verified to produce a *p-operator space* structure on \mathcal{X} which is minimal in the sense that $\|\cdot\|_n \leq \|\cdot\|'_n$ (for each n) with any other operator space structure on \mathcal{X} . We call this operator space structure the *minimal p-operator space* structure on \mathcal{X} . If \mathcal{V} is an operator space and $T : \mathcal{V} \rightarrow \mathcal{X}$ is bounded then T is completely bounded with $\|T\|_{\text{pcb}} = \|T\|$. Then, by uniformity of the injective tensor product, we see that

$$T^{(n)} \cong (T \otimes \text{id}) \circ \iota_n : \mathcal{V} \check{\otimes}^p \mathcal{B}(\ell^p(n)) \rightarrow \mathcal{V} \otimes^\lambda \mathcal{B}(\ell^p(n)) \rightarrow \mathcal{X} \otimes^\lambda \mathcal{B}(\ell^p(n))$$

is bounded with norm at most $\|T\|$, where ι_n is the identity on $\mathcal{V} \otimes \mathcal{B}(\ell^p(n))$, which is a contraction as $\check{\otimes}^p$ gives a reasonable cross-norm. We call any *p-operator space* \mathcal{V} whose *p-operator structure* is the minimal one, i.e. $\mathcal{V} = \min \mathcal{V}$ completely isometrically, a *minimal p-operator space*.

Proposition 0.1. *The following are equivalent for a p-operator space \mathcal{V} :*

- (i) \mathcal{V} is minimal;

- (ii) for any p -operator space \mathcal{W} , $\mathcal{CB}_p(\mathcal{W}, \mathcal{V}) = \mathcal{B}(\mathcal{W}, \mathcal{V})$ isometrically;
- (iii) for any p -operator space \mathcal{W} , $\mathcal{W}^{\otimes p} \mathcal{V} = \mathcal{W} \otimes^\lambda \mathcal{V}$.

Proof. Since $\mathcal{CB}_p(\mathcal{W}, \mathcal{V}) \subset \mathcal{B}(\mathcal{W}, \mathcal{V})$ contractively, the observation above gives that (i) implies (ii). Condition (ii) implies that $\text{id} : \min \mathcal{V} \rightarrow \mathcal{V}$ is completely contractive. Since the converse is automatic, (i) holds.

If (ii) holds then $\mathcal{CB}_p(\mathcal{W}^*, \mathcal{V}) = \mathcal{B}(\mathcal{W}^*, \mathcal{V})$ isometrically. Thus, by virtue of the definition of the p -operator injective tensor product ([1, §3]) and the well known injection $\mathcal{W} \otimes \mathcal{V} \hookrightarrow \mathcal{B}(\mathcal{W}^*, \mathcal{V})$, the p -operator injective and injective tensor norms agree on $\mathcal{W} \otimes \mathcal{V}$. \square

The definition of maximal p -operator space will be given in Section 2.

The following rudimentary fact will be referred to a couple of times in the sequel, and is an obvious consequence of the density of simple functions in $L^{p'}(\phi)$ and duality.

Lemma 0.2. *For any finite subset $F \subset L^p(\phi)$ and $\varepsilon > 0$, there is an m in \mathbb{N} and a contraction $V : L^p(\phi) \rightarrow \ell^p(m)$ for which $(1 - \varepsilon) \|f\|_{L^p} \leq \|Vf\|_{\ell^p} \leq (1 + \varepsilon) \|f\|_{L^p}$ for f in F .*

1. ON MINIMAL p -OPERATOR SPACES

In the theory of 2-operator spaces, a special role is played by commutative C^* -algebras and completely isometric copies of their subspaces. These are the *minimal operator spaces*. Classical theory tells us that any representation of a commutative C^* -algebra $\mathcal{A} \cong C_0(\Omega)$ on a Hilbert space can be realised as a direct sum of representations on cyclic subspaces, where each, in turn, produces a Radon measure ν on Ω by which the representation is unitarily equivalent to a representation by multiplication operators on $L^2(\nu)$. We are not aware of any analogue of this result for representation on \mathcal{SQ}_p -spaces, or even L^p -spaces. This reduces us to studying representations which are already multiplication representations on L^p -spaces. This gives rise to a more robust theory than might be anticipated.

1.1. On the space of continuous functions as a minimal p -operator space.

We begin with the continuous bounded functions $\mathcal{C}_b(\Omega)$ on a locally compact space Ω . In this case a familiar formula for the injective tensor product gives for each n an isometric identification

$$(1.1) \quad \mathbb{M}_n(\min \mathcal{C}_b(\Omega)) \cong \mathcal{C}_b(\Omega, \mathcal{B}(\ell^n)), \quad [f_{ij}] \mapsto (\omega \mapsto [f_{ij}(\omega)]).$$

Indeed, the Stone-Čech compactification satisfies $\mathcal{C}(\beta\Omega, M) \cong \mathcal{C}_b(\Omega, M)$ for any finite dimensional Banach space M . We let ν be a Radon measure on Ω and $M_\nu : \mathcal{C}_b(\Omega) \rightarrow \mathcal{B}(L^p(\nu))$ be the contractive injection given by

$$M_\nu(f)\xi(\omega) = f(\omega)\xi(\omega)$$

for ν -a.e. ω . We say ν is faithful if $\nu(U) > 0$ for any open set U . If ν is faithful then M_ν is an isometry.

The next simple result is required for the next section. The result seems like it ought to hold for more general L^∞ -spaces, except for a certain localisation of norm argument at the end of the proof.

Proposition 1.1. *Given a faithful Radon measure ν on Ω , $M_\nu : \min \mathcal{C}_b(\Omega) \rightarrow \mathcal{B}(L^p(\nu))$ is a complete isometry.*

Proof. It suffices to verify that each amplification $M_\nu^{(n)}$ is an isometry. We identify $\mathbb{M}_n(\mathcal{B}(L^p(\nu))) \cong \mathcal{B}(L^p(\nu, \ell^p(n)))$. We observe, under this identification, that $M_\nu^{(n)}(F)\xi(\omega) = F(\omega)\xi(\omega)$, for F in $\mathcal{C}_b(\Omega, \mathcal{B}(\ell^p(n)))$, ξ in $L^p(\nu, \ell^p(n))$ and ν -a.e. ω . We compute

$$\begin{aligned} \|M_\nu^{(n)}(F)\xi\|_{L^p(\nu, \ell^p)} &= \left(\int_\Omega \|F(\omega)\xi(\omega)\|_{\ell^p}^p d\nu(\omega) \right)^{1/p} \\ &\leq \left(\int_\Omega \|F(\omega)\|_{\mathcal{B}(\ell^p)}^p \|\xi(\omega)\|_{\ell^p}^p d\nu(\omega) \right)^{1/p} \\ &\leq \|F\|_{\mathcal{C}_b(\Omega, \mathcal{B}(\ell^p))} \|\xi\|_{L^p(\nu, \ell^p)} \end{aligned}$$

Thus $M_\nu^{(n)}$ is a contraction.

Conversely, given $\varepsilon > 0$, find ω_0 for which $\|F(\omega_0)\|_{\mathcal{B}(\ell^p)} > \|F\|_{\mathcal{C}_b(\Omega, \mathcal{B}(\ell^p))} - \varepsilon$, and then ξ_0 in $\ell^p(n)$ with $\|\xi_0\|_{\ell^p} = 1$ and for which $\|F(\omega_0)\xi_0\|_{\ell^p} = \|F(\omega_0)\|_{\mathcal{B}(\ell^p)}$. Find a compact neighbourhood K of ω_0 such that $\|F(\omega) - F(\omega_0)\|_{\mathcal{B}(\ell^p)} < \varepsilon$ for ω in K . (This is the “localisation of norm argument” to which we alluded, above.) Then $\xi = \nu(K)^{-1/p} 1_K(\cdot)\xi_0$ in $L^p(\nu, \ell^p(n))$ is of norm 1 and satisfies

$$\|M_\nu^{(n)}(F)\xi - F(\omega_0)\xi\|_{L^p(\nu, \ell^p)} < \varepsilon.$$

It is immediate that $M_\nu^{(n)}$ is an isometry. \square

Of course, the above result applies to $\ell^\infty(\Omega)$ for any set Ω . Let \mathcal{X} be a Banach space. We let Ω denote any subset of the unit ball of \mathcal{X}^* which is norming for \mathcal{X} , and consider the isometric embedding

$$(1.2) \quad \mathcal{X} \hookrightarrow \ell^\infty(\Omega), \quad x \mapsto (\omega \mapsto \omega(x)).$$

As already observed in [10], this is a complete isometry of minimal spaces, hence $\min \mathcal{X}$ acts on L^p .

1.2. L^∞ as a minimal p -operator space. We show that for a suitable measure μ , $L^\infty(\mu)$ attains its minimal p -operator space structure as multiplication operators on $L^p(\mu)$.

We say a measure μ is decomposable if we can write $\mu = \sum_{i \in I} \mu_i$ where each μ_i is finite, and μ_i and $\mu_{i'}$ are mutually singular for distinct indices. For such measures, we have the duality $L^1(\mu)^* \cong L^\infty(\mu)$, provided we define $L^\infty(\mu)$ to be certain equivalence classes of locally essentially bounded functions; see [9, p. 192].

We will hereafter assume μ is a decomposable measure.

We require a certain p -analogue of a familiar result in representation theory of commutative C^* -algebras holds; see [4, II.1.1], for example, whose standard proof we modify. We let $M_\mu : L^\infty(\mu) \rightarrow \mathcal{B}(L^p(\mu))$ be the representation given by multiplication operators.

Lemma 1.2. *There is a locally compact space Ω such that $L^\infty(\mu) \cong \mathcal{C}_b(\Omega)$ via a $*$ -algebra isomorphism $f \mapsto \hat{f}$, a faithful Radon measure ν on Ω , and a surjective isometry $U : L^p(\nu) \rightarrow L^p(\mu)$ such that $UM_\nu(\hat{f}) = M_\mu(f)U$.*

Proof. We first assume that μ is finite. (The proof will work for the σ -finite case as well.) In this case there is a norm 1 cyclic and separating vector ξ for M_μ ; indeed, let

ξ be any fully supported norm one element. We let Ω denote the Gelfand spectrum of $L^\infty(\mu)$ and $f \mapsto \hat{f}$ the Gelfand transform. We observe that $|\widehat{f}|^p = |\hat{f}|^p$.

We define ν on Ω by

$$\int_{\Omega} \hat{f} d\nu = \int f |\xi|^p d\mu.$$

Since ξ is fully supported, ν is faithful. We then define $U : \mathcal{C}(\Omega) \rightarrow L^p(\mu)$ by $U\hat{f} = f\xi$. We observe that

$$\|U\hat{f}\|_{L^p(\mu)}^p = \int |f|^p |\xi|^p d\mu = \int |\hat{f}|^p d\nu = \|\hat{f}\|_{L^p(\nu)}^p.$$

Since $\mathcal{C}(\Omega)$ is dense in $L^p(\nu)$, and ξ is a cyclic vector, U extends to a surjective isometry on $L^p(\nu)$. Finally, if $f, g \in L^\infty(\mu)$, then

$$UM_\nu(\hat{f})\hat{g} = U\widehat{fg} = fg\xi = M_\mu(f)U\hat{g}$$

which, again by density of $\mathcal{C}(\Omega)$ in $L^p(\nu)$, shows that $UM_\nu(\hat{f}) = M_\mu(f)U$.

Now consider general decomposable $\mu = \sum_{\iota \in I} \mu_\iota$. Let for each ι , Ω_ι denote the Gelfand spectrum of $L^\infty(\mu_\iota)$ and we have C^* -isomorphisms

$$L^\infty(\mu) \cong \ell^\infty\text{-}\bigoplus_{\iota \in I} L^\infty(\mu_\iota) \cong \ell^\infty\text{-}\bigoplus_{\iota \in I} \mathcal{C}(\Omega_\iota) \cong \mathcal{C}_b(\Omega)$$

where $\Omega = \bigsqcup_{\iota \in I} \Omega_\iota$ is the topological coproduct. Let $f \mapsto \hat{f}$ denote the composite isomorphism. We observe, moreover, that $L^p(\mu) \cong \ell^p\text{-}\bigoplus_{\iota \in I} L^p(\mu_\iota)$, where each $L^p(\mu_\iota)$ is an M_μ -invariant subspace. We let ν_ι be a measure supported on Ω_ι given as above, and $U_\iota : L^p(\nu_\iota) \rightarrow L^p(\mu_\iota)$ the associated surjective isometry intertwining $M_{\mu_\iota} = M_\mu|_{L^p(\mu_\iota)}$ and M_{ν_ι} . Then $U = \bigoplus_{\iota \in I} U_\iota$ is the desired isometry intertwining M_μ and M_ν . \square

Theorem 1.3. *The map $M_\mu : \min L^\infty(\mu) \rightarrow \mathcal{B}(L^p(\mu))$ is a complete isometry.*

Proof. The above lemma provides a map $f \mapsto \hat{f} : L^\infty(\mu) \rightarrow \mathcal{C}_b(\Omega)$, which is a complete isometry for the minimal p -operator space structure on both spaces, a faithful Radon measure ν on Ω and a surjective isometry $U : L^p(\nu) \rightarrow L^p(\mu)$ such that $M_\mu(f) = U^{-1}M_\nu(\hat{f})U$. Since M_ν is completely isometric by Proposition 1.1, we find that M_μ is a complete isometry. \square

On the topic of $L^\infty(\mu)$, we record the following useful result, aspects of which are folklore. This will be used in Section 2.3.

Lemma 1.4. (i) *$M_\mu(L^\infty(\mu))$ is its own commutant in $\mathcal{B}(L^p(\mu))$, and hence a weak*-closed subalgebra.*

(ii) *There is a completely contractive expectation $E : \mathcal{B}(L^p(\mu)) \rightarrow M_\mu(L^\infty(\mu))$, i.e. $E(M_\mu(f)TM_\mu(g)) = M_\mu(f)E(T)M_\mu(g)$ for f, g in $L^\infty(\mu)$ and T in $\mathcal{B}(L^p(\mu))$.*

Proof. (i) Let \mathcal{F} be the family of μ -finite sets. If $F \in \mathcal{F}$ then $1_F \in L^\infty \cap L^p(\mu)$. Fix T in the comutant of $M_\mu(L^\infty(\mu))$ in $\mathcal{B}(L^p(\mu))$ and let $h_F = T1_F$ for F in \mathcal{F} . We observe that for ξ in $L^\infty \cap L^p(\mu)$, the space of which is dense in $L^p(\mu)$, that $T(1_F\xi) = T(1_F)\xi = h_F\xi$, from which it easily follows that $h_F \in L^\infty(\mu)$ with $\|h_F\|_\infty \leq \|T\|$. It is clear that $1_F h_{F'} = 0$ and $h_F + h_{F'} = h_{F \cup F'}$, if $F \cap F'$ is μ -null. We let $\{F_\iota\}_{\iota \in I}$ be a family of sets witnessing the decomposability of μ . We observe that the net $(\sum_{\iota \in J} h_{F_\iota})_J$, indexed over the increasing family of finite subsets of I ,

converges weak* to an element h of $L^\infty(\mu)$. Indeed if $\psi \in L^1(\mu)$ then there is a σ -finite set S so $1_S\psi = \psi$ and

$$\lim_J \int \sum_{\iota \in J} h_{F_\iota} \psi d\mu = \int \sum_{\iota \in I_S} h_{F_\iota} \psi d\mu$$

where $I_S = \{\iota : \mu(F_\iota \cap S) > 0\}$ is countable. In particular, $h\psi = \sum_{\iota \in I_S} h_{F_\iota} \psi$. Now if $\xi \in L^p \cap L^\infty(\mu)$ and $\eta \in L^{p'}(\mu)$ we let S be σ -finite so $1_S\xi = \xi$ and we have

$$\begin{aligned} \int (T\xi)\eta d\mu &= \int T\left(\sum_{\iota \in I_S} 1_{F_\iota}\xi\right)\eta d\mu = \int \sum_{\iota \in I_S} T(1_{F_\iota}\xi)\eta d\mu \\ &= \int \sum_{\iota \in I_S} h_{F_\iota}\xi\eta d\mu = \int h\xi\eta d\mu. \end{aligned}$$

Thus $T = M_\mu(h)$. The commutant of any set in $\mathcal{B}(L^p(\mu))$ is weak*-closed.

(ii) We let $U^\infty(\mu) = \{u \in L^\infty(\mu) : u^*u = 1\}$. Let m be an invariant mean on $\ell^\infty(U^\infty(\mu))$, which we may consider, notationally, as a finitely additive measure. We define E by

$$E(T) = \int_{U^\infty(\mu)} M_\mu(u) T M_\mu(u^*) dm(u)$$

where the “integral” is understood in the weak* sense. Since $\text{span} U^\infty(\mu) = L^\infty(\mu)$, it is immediate that E is a contractive expectation. If $T \in \mathbb{M}_n(\mathcal{B}(L^p(\mu))) \cong \mathcal{B}(\ell^p(n) \otimes^p L^p(\mu))$ we observe that

$$E^{(n)}(T) = \int_{U^\infty(\mu)} (I \otimes M_\mu(u)) T (I \otimes M_\mu(u^*)) dm(u)$$

Hence E is completely contractive. \square

2. MAXIMAL p -OPERATOR SPACES

2.1. Definitions and basic properties. For a Banach space \mathcal{X} , we consider two p -operator space structures on \mathcal{X} , whose norms on x in $\mathbb{M}_n(\mathcal{X})$ are given by

$$\begin{aligned} \|x\|_{\max_{L^p}} &= \sup \left\{ \left\| \pi^{(n)}(x) \right\| : \pi : \mathcal{X} \rightarrow \mathcal{B}(L^p(\phi)) \text{ is a contraction, } \phi \text{ is a measure} \right\} \\ &= \sup \left\{ \left\| \pi^{(n)}(x) \right\| : \pi : \mathcal{X} \rightarrow \mathcal{B}(\ell^p(m)) \text{ is a contraction, } m \in \mathbb{N} \right\} \\ \|x\|_{\max} &= \sup \left\{ \left\| \pi^{(n)}(x) \right\| : \pi : \mathcal{X} \rightarrow \mathcal{B}(E) \text{ is a contraction, } E \in \mathcal{SQ}_p \right\}. \end{aligned}$$

The equality of the two descriptions of $\|\cdot\|_{\max_{L^p}}$ is an immediate consequence of Lemma 0.2. It is clear that these norms give p -operator space structures on \mathcal{X} , which we call the *maximal structure on L^p* and the *maximal structure*, respectively. We denote the associated operator spaces by $\max_{L^p} \mathcal{X}$ and $\max \mathcal{X}$. There is an equivalent formulation of $\max \mathcal{X}$ given in [10, p. 95], presented in a local context. It is clear that $\text{id} : \max \mathcal{X} \rightarrow \max_{L^p} \mathcal{X}$ is a complete contraction. There is no loss of generality if we replace contractions π , above, by isometries; simply consider the isometry $\text{id} : \mathcal{X} \rightarrow \min \mathcal{X}$ which acts on L^p by (1.2).

It is clear that for every operator space \mathcal{V} and v in $\mathbb{M}_n(\mathcal{V})$, that

$$\|v\| \leq \|v\|_{\max}.$$

It is unknown to the authors whether the operator space structures \max and \max_{L^p} coincide on any non-trivial Banach space. We thus use the following definition. We say that an p -operator space \mathcal{V} is of *maximal type* if for v in $\mathbb{M}_n(\mathcal{V})$ we have

$$\|v\|_{\max_{L^p}} \leq \|v\|$$

Lemma 2.1. *Let \mathcal{V} be a p -operator space. Then the following are equivalent:*

- (i) \mathcal{V} is of maximal type;
- (ii) $\mathcal{CB}_p(\mathcal{V}, \mathcal{Z}) = \mathcal{B}(\mathcal{V}, \mathcal{Z})$ isometrically for any p -operator space \mathcal{Z} acting on L^p ;
- (iii) $\mathcal{CB}_p(\mathcal{V}, \mathcal{B}(\ell^p(n))) = \mathcal{B}(\mathcal{V}, \mathcal{B}(\ell^p(n)))$ isometrically for each n .

Proof. It is the case for any operator space \mathcal{V} that $\mathcal{CB}_p(\mathcal{V}, \mathcal{B}(\ell^p(n))) \subset \mathcal{B}(\mathcal{V}, \mathcal{B}(\ell^p(n)))$ contractively. We obtain the converse inclusion, contractively, only for maximal type p -operator spaces, by definition. Thus (i) is equivalent to (ii). That (ii) implies (iii) is obvious. That (iii) implies (ii) is a consequence of Lemma 0.2. \square

Corollary 2.2. *Let \mathcal{V} be a p -operator space. Then the following are equivalent:*

- (i') \mathcal{V} is of maximal type;
- (ii') $\mathcal{V} \hat{\otimes}^p \mathcal{W} = \mathcal{V} \otimes^\gamma \mathcal{W}$, isometrically, for any p -operator space \mathcal{W} ;
- (iii') $\mathcal{V} \hat{\otimes}^p \mathcal{N}(\ell^p(m)) = \mathcal{V} \otimes^\gamma \mathcal{N}(\ell^p(m))$, isometrically, for any m .

Proof. We will show that each statement (n') of the present result, is equivalent to statment (n) of Lemma 2.1

We have that \mathcal{W}^* represents completely isometrically on some L^p by [5, Thm. 4.3]. Hence, thanks to the well-known dual paring $\langle v \otimes w, T \rangle = Tv(w)$ of $\mathcal{V} \otimes \mathcal{W}$ with $\mathcal{B}(\mathcal{V}, \mathcal{W}^*)$ and its p -operator space analogue ([5, Prop. 4.9]), if (ii) of the above lemma holds, then the p -operator projective and projective tensor norms agree on $\mathcal{V} \otimes \mathcal{W}$. If (ii') holds, then statment (ii) of the above lemma holds whenever $\mathcal{Z} = \mathcal{W}^*$, i.e. for any p -operator dual space. Hence statment (ii) holds with \mathcal{Z}^{**} in place of \mathcal{Z} we let $\kappa_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{Z}^{**}$ denote the canonical embedding and have that $\mathcal{B}(\mathcal{V}, \mathcal{Z}) \cong \kappa_{\mathcal{Z}} \circ \mathcal{B}(\mathcal{V}, \mathcal{Z}) \subset \mathcal{B}(\mathcal{V}, \mathcal{Z}^{**}) = \mathcal{CB}_p(\mathcal{V}, \mathcal{Z}^{**})$ isometrically. If \mathcal{Z} acts on L^p then, by [5, Prop. 4.4], $\kappa_{\mathcal{Z}}$ is a complete isometry so $\mathcal{CB}_p(\mathcal{V}, \mathcal{Z}) \cong \kappa_{\mathcal{Z}} \circ \mathcal{CB}_p(\mathcal{V}, \mathcal{Z}) \subset \mathcal{CB}_p(\mathcal{V}, \mathcal{Z}^{**})$ isometrically, hence $\mathcal{B}(\mathcal{V}, \mathcal{Z}) \cong \kappa_{\mathcal{Z}} \circ \mathcal{B}(\mathcal{V}, \mathcal{Z}) = \kappa_{\mathcal{Z}} \circ \mathcal{CB}_p(\mathcal{V}, \mathcal{Z}) \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{Z})$ isometrically, hence statment (ii) holds generally.

Just as above, (iii') holds if and only if (iii) of the above lemma holds. \square

We observe that if \mathcal{V} and \mathcal{W} are each maximal type p -operator spaces, then $\mathcal{V} \hat{\otimes}^p \mathcal{W}$ is also of maximal type. Indeed, if \mathcal{Z} acts on L^p , [5, Prop. 4.9] provided isometric identifications

$$\mathcal{CB}_p(\mathcal{V} \hat{\otimes}^p \mathcal{W}, \mathcal{Z}) \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{CB}_p(\mathcal{W}, \mathcal{Z})) = \mathcal{B}(\mathcal{V}, \mathcal{B}(\mathcal{W}, \mathcal{Z})) = \mathcal{B}(\mathcal{V} \otimes^\gamma \mathcal{W}, \mathcal{Z})$$

and we appeal to statements (ii) and (ii') above. We are unaware of whether $\max_{L^p} \mathcal{V} \hat{\otimes}^p \max_{L^p} \mathcal{W}$ is completely isometric to $\max_{L^p}(\mathcal{V} \otimes^\gamma \mathcal{W})$, but this does hold for L^1 -spaces, as we will see in Section 2.3

2.2. Duality and quotients.

Proposition 2.3. (i) *If \mathcal{V} is a maximal type p -operator space then the dual structure is minimal, i.e. $\mathcal{V}^* = \min \mathcal{V}^*$. In particular, $(\max \mathcal{V})^* = \min \mathcal{V}^* = (\max_{L^p} \mathcal{V})^*$.*

(ii) *If \mathcal{V} is a complete quotient of a maximal type p -operator space, then \mathcal{V} is of maximal type.*

Proof. (i) We follow the proof form classical operator spaces – see [2, Cor. 2.8] or [7, (3.3.13)] – and use Lemma 2.1. We let Ω be a dense subset of the unit ball of \mathcal{V} , and we have complete isometries

$$\mathbb{M}_n(\mathcal{V}^*) \cong \mathcal{CB}(\mathcal{V}, \mathcal{B}(\ell^p(n))) = \mathcal{B}(\mathcal{V}, \mathcal{B}(\ell^p(n))) \tilde{\subset} \ell^\infty(\Omega, \mathcal{B}(\ell^p(n)))$$

whose composition is given by $[\psi_{ij}] \mapsto (\omega \mapsto [\psi_{ij}(\omega)])$. By (1.2) this is the minimal p -operator structure on \mathcal{V}^* .

(ii) If $q : \mathcal{V} \rightarrow \mathcal{Z}$ is a complete quotient map, and $T : \mathcal{Z} \rightarrow \mathcal{B}(\ell^p(n))$ is a linear contraction, then $T \circ q : \mathcal{V} \rightarrow \mathcal{B}(\ell^p(n))$ is a contraction, hence a complete contraction by (i). Thus if z is in the open unit ball of $\mathbb{M}_n(\mathcal{Z})$, there is v in the open unit ball of $\mathbb{M}_n(\mathcal{V})$ so $z = q^{(n)}(v)$. Then for any linear contraction $T : \mathcal{Z} \rightarrow \mathcal{B}(\ell^p(n))$ we have $\|T^{(n)}(z)\|_{\mathcal{B}(\ell^p)} = \|(T \circ q)^{(n)}(v)\|_{\mathcal{B}(\ell^p)} < 1$, so T is a complete contraction. \square

We aim obtain the dual statment to (i), above. We note that unlike in the 2-operator space setting, it is not a priori obvious that $(\min \mathcal{C}(\Omega))^{**} = \min \mathcal{C}(\Omega)^{**}$ completely isometrically, though we will establish this fact below.

We require a preparatory idea from the theory of vector measures. For a compact Hausdorff space Ω we let $M(\Omega)$ denote the space of complex Borel measures on Ω . Furthermore, if E is a Banach space we let $M(\Omega, E)$ denote the E -valued Borel measures on Ω of bounded variation. If E satisfies the Radon-Nikodym property of [8, p. 61] we have

$$(2.1) \quad M(\Omega, E) = \bigcup_{\nu \in M^+(\Omega)} L^1(\nu, E) \cong \bigcup_{\nu \in M^+(\Omega)} L^1(\nu) \otimes^\gamma E \cong M(\Omega) \otimes^\gamma E$$

where the implied isomorphism is isometric. Indeed, if $G \in M(\Omega, E)$, there is ν in $M^+(\Omega)$ and g in $L^1(\nu, E)$ for which $G(B) = \int_B g d\nu$, with $\|G\|_{M(\Omega, E)} = |G|(B) = \|g\|_{L^1(\nu, E)}$. It is well-known that $L^1(\nu, E) \cong L^1(\nu) \otimes^\gamma E$ isometrically. Since, by Lebesgue decomposition, $L^1(\nu)$ is contractively compelmmented in $M(\Omega)$, we have that $L^1(\nu) \otimes^\gamma E$ embeds isometrically into $M(\Omega) \otimes^\gamma E$. Moreover, each element in $M(\Omega) \otimes^\gamma E$ is an element of some $L^1(\nu) \otimes^\gamma E$. Indeed, write an element of the former as $\sum_{k=1}^\infty \nu_k \otimes x_k$, where each $\|x_k\|_E = 1$ and $\sum_{k=1}^\infty \|\nu_k\|_M < \infty$. Then let $\nu = \sum_{k=1}^\infty |\nu_k|$ and observe that each $\nu_k \ll \nu$, so the element is in $L^1(\nu) \otimes^\gamma E$.

Theorem 2.4. *If \mathcal{W} is a minimal operator space, then its dual operator space is maximal on L^p , i.e. $(\min \mathcal{W})^* = \max_{L^p} \mathcal{W}^*$.*

Proof. We begin with $\min \mathcal{C}(\Omega)$ for a compact space. From the formula $\mathcal{V} \tilde{\otimes}^p \mathcal{B}(\ell^p(n)) \cong \mathbb{M}_n(\mathcal{V})$ on one hand, and then (1.1) on the other, we obtain for each n , isometric identifications

$$\min \mathcal{C}(\Omega) \tilde{\otimes}^p \mathcal{B}(\ell^p(n)) \cong \mathbb{M}_n(\min \mathcal{C}(\Omega)) \cong \mathcal{C}(\Omega, \mathcal{B}(\ell^p(n))).$$

Taking duals, we have from [1, Theo. 3.6] on one hand, and [15, 16] (or see [8, p. 182]) on the other, that

$$(\min \mathcal{C}(\Omega))^* \widehat{\otimes}^p \mathcal{N}(\ell^p(n)) \cong M(\Omega, \mathcal{N}(\ell^p(n))).$$

Thanks to the fact that finite dimensional spaces enjoy the Radon-Nikodym property, we can use (2.1) on the right hand side of the above identification to see that

$$(\min \mathcal{C}(\Omega))^* \widehat{\otimes}^p \mathcal{N}(\ell^p(n)) = M(\Omega) \otimes^\gamma \mathcal{N}(\ell^p(n))$$

isometrically for each n . By Corollary 2.2 we see that $M(\Omega)$, in its capacity as the dual of $\min \mathcal{C}(\Omega)$, admits a maximal type p -operator space structure. Since this is a dual space, it follows [5, Thm. 4.3] that this is the maximal structure on L^p .

Now we consider $\min \mathcal{W} \tilde{\subset} \min \mathcal{C}(\Omega)$ where Ω is the unit ball of \mathcal{W}^* with weak* topology. Hence $\mathcal{W}^* \cong \max_{L^p} M(\Omega) / \{\nu : \langle \nu, w \rangle = 0 \text{ for } w \in \mathcal{W}\}$ completely isometrically. By (iii) of Proposition 2.3, \mathcal{W}^* is of p -maximal type. But by [5, Thm. 4.3], \mathcal{W}^* acts on some L^p , hence the operator space structure is \max_{L^p} . \square

We observe that it is an immediate consequence of Theorem 2.4 and Proposition 2.3 (i), that

$$(2.2) \quad (\min \mathcal{V})^{**} = \min \mathcal{V}^{**}$$

completely isometrically.

As another consequence we see that for any Banach space \mathcal{X} and \mathcal{Y}

$$(2.3) \quad \min \mathcal{X} \check{\otimes}^p \min \mathcal{Y} = \min(\mathcal{X} \otimes^\lambda \mathcal{Y})$$

completely isometrically. Indeed, we have from Lemma 2.1 (ii) and (0.1), that

$$\begin{aligned} \mathbb{M}_n(\mathcal{CB}_p(\max \mathcal{X}^*, \min \mathcal{Y})) &\cong \mathcal{CB}_p(\max \mathcal{X}^*, \mathbb{M}_n(\min \mathcal{Y})) \\ &= \mathcal{B}(\mathcal{X}^*, \mathbb{M}_n(\min \mathcal{Y})) \cong \mathcal{B}(\mathcal{X}^*, \mathcal{Y} \otimes^\lambda \mathcal{B}(\ell^p(n))) \end{aligned}$$

isometrically. Thus the embedding of $\mathbb{M}_n(\mathcal{X} \otimes \mathcal{Y}) \cong \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{B}(\ell^p(n))$ into the space above establishes that

$$\mathbb{M}_n(\min \mathcal{X} \check{\otimes}^p \min \mathcal{Y}) = \mathcal{X} \otimes^\lambda \mathcal{Y} \otimes^\lambda \mathcal{B}(\ell^p(n))$$

isometrically, for each n . Then (2.3) follows from (0.1).

2.3. L^1 spaces. Spaces $L^1(\mu)$, for a decomposable measure μ , are the most natural class of maximal p -operator spaces.

Theorem 2.5. *The operator space structure on $L^1(\mu)$, as a subspace of $(\min L^\infty(\mu))^*$, is the maximal structure on L^p , i.e. $\max_{L^p} L^1(\mu)$.*

Proof. We will establish that with the operator space structure given by $L^1(\mu) \hookrightarrow (\min L^\infty(\mu))^*$, we have $\mathcal{CB}_p(L^1(\mu), \mathcal{V}) = \mathcal{B}(L^1(\mu), \mathcal{V})$ isometrically, for any p -operator space \mathcal{V} acting on some L^p . By Lemma 2.1, this implies that $L^1(\mu)$ is of maximal type. However, since $(\min L^\infty(\mu))^*$ acts on L^p ([5, Thm. 4.3]), this is the \max_{L^p} structure.

The assumption that \mathcal{V} acts on L^p implies that the embedding $\kappa_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}^{**}$ is a complete isometry ([5, Prop. 4.4]). We also note that

$$(2.4) \quad L^1(\mu)^* \cong \min L^\infty(\mu) \text{ completely isometrically.}$$

Indeed, as noted in [11, Prop. 1.6.13], it is sufficient, by virtue of [5, Prop. 5.5] to observe that $\min L^\infty(\mu) \cong M_\mu(L^\infty(\mu))$ is weak* closed. This was shown in Lemma 1.4.

We consider, first, the adjoint $S^* : \mathcal{V}^* \rightarrow L^1(\Omega)^* \cong L^\infty(\Omega)$, which is completely bounded with $\|S^*\|_{\text{pcb}} = \|S^*\| = \|S\|$ by Proposition 0.1 (ii). We then have that $S = S^{**} \circ \kappa_{L^1(\mu)} : L^1(\mu) \rightarrow \kappa_{\mathcal{V}}(\mathcal{V}) \cong \mathcal{V}$ satisfies $\|S\|_{\text{pcb}} \leq \|S^{**}\|_{\text{pcb}}$, which, by [5, Lem. 4.5], is no greater than $\|S^*\|_{\text{pcb}} = \|S\|$. \square

The following is an immediate consequence of Lemma 1.4 and [5, Prop. 5.6].

Corollary 2.6. *The map $\eta \otimes \xi \mapsto \eta\xi$ extends to a complete quotient map from $\mathcal{N}(L^p(\mu)) = L^{p'}(\mu) \otimes^\gamma L^p(\mu)$ onto $\max_{L^p} L^1(\mu)$.*

We obtain the following useful tensor product formulas. If \mathcal{V} is a p -operator space, Corollary 2.2 provides the isometric identifications

$$\max_{L^p} L^1(\mu) \widehat{\otimes}^p \mathcal{V} = L^1(\mu) \otimes^\gamma \mathcal{V} \cong L^1(\mu, \mathcal{V}).$$

Also we obtain a completely isometric identification

$$(2.5) \quad \max_{L^p} L^1(\mu) \widehat{\otimes}^p \max_{L^p} L^1(\nu) = \max_{L^p} (L^1(\mu) \otimes^\gamma L^1(\nu)) \cong \max_{L^p} L^1(\mu \times \nu).$$

Indeed we have an isometric identification $\max_{L^p} L^1(\mu) \widehat{\otimes}^p \max_{L^p} L^1(\nu) = L^1(\mu) \otimes^\gamma L^1(\nu) \cong L^1(\mu \times \nu)$. The first space has dual $\min L^\infty(\mu) \overline{\otimes}_F \min L^\infty(\nu)$ (Fubini product) in $\mathcal{B}(L^p(\mu) \otimes^p L^p(\nu))$ by [5, Thm. 6.3], while the third has dual $L^\infty(\mu \times \nu)$. The latter space acts as multiplication operators on $L^p(\mu \times \nu) \cong L^p(\nu) \otimes^p L^p(\mu)$. This dual identification shows that $\min L^\infty(\mu) \overline{\otimes}_F \min L^\infty(\nu) \cong \min L^\infty(\mu \times \nu)$. Hence (2.5) follows.

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